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RETARDATION OF A CRACK WITH CONNECTIONS BETWEEN THE FACES USING AN INDUCED THERMOELASTIC STRESS FIELD

R. I. Kadiev and V. M. Mirsalimov

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This paper considers local temperature variations near the tip of a crack in the presence of regions in which the crack faces interact. It is assumed that these regions are adjacent to the crack tip and are comparable in size to the crack size. The problem of local temperature variations consists of delay or retardation of crack growth. For a crack with connections between the crack faces subjected to external tensile loads, an induced thermoelastic stress field, and the stresses at the connections preventing crack opening, the boundary-value problem of the equilibrium of the crack reduces to a system of nonlinear singular integrodifferential equations with a Cauchy kernel. The normal and tangential stresses at the connections are found by solving this system of equations. The stress intensity factors are calculated. The energy characteristics of cracks with tip regions are considered. The limiting equilibrium condition for cracks with tip regions is formulated using the criterion of limiting stretching of the connections.

Key words: *cracks, thermoelastic stress field.*

Formulation of the Problem. Designing reliable emergency response systems is a vital problem, especially as far as unique facilities and human safety are concerned. One of the effective means for retarding crack growth are temperature and thermoelastic fields [1, 2]. In fracture mechanics, the problem of crack healing is of great importance. The results of [3] show that the effect of a heat source reduces the strain of an extended plane in a direction perpendicular to the crack, and, hence, the stress intensity factor in the neighborhood of the crack tip decreases.

Let us consider an unbounded elastic plane with one rectilinear crack of length $2l$ at the coordinate origin. We assume the presence of regions in which the crack faces interact so that this interaction retards crack opening. It is assumed that these regions are adjacent to the crack tip and their sizes are comparable to the crack length.

Outside the tip regions, the crack faces are free of external loads. At infinity, the plane is subjected to uniform extension along the ordinate $\sigma_y^\infty = \sigma_0$ (Fig. 1).

The crack propagation is retarded by producing a zone of compressing stresses on the crack propagation path using a heat source which heats a region S to a temperature T_0.

The following assumptions are adopted:

(a) all thermoelastic characteristics of the material of the plane do not depend on temperature;

(b) the material of the plane is a homogeneous and isotropic body.

It is assumed that at the time $t = 0$, an arbitrary region S on the crack propagation path in the plane is heated instantaneously to a constant temperature $T = T_0$. The remaining part of the plane has zero temperature at the initial time.

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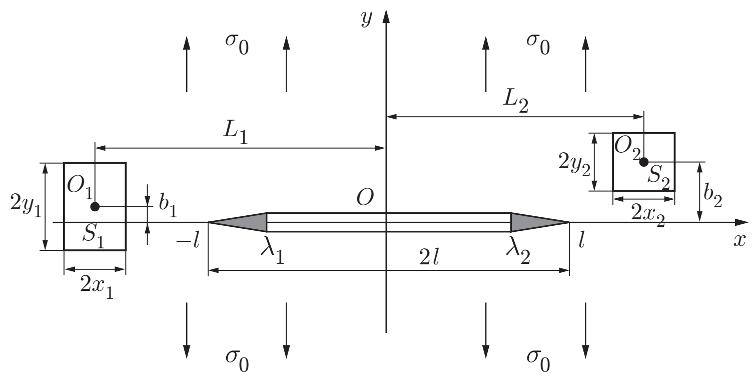

Fig. 1. Computational diagram of the problem.

We distinguish crack regions of length d_1 and d_2 (tip regions) adjacent to its tip in which the crack faces interact. The interaction of the crack faces in the tip regions is modeled by introducing connections (adhesive forces) between crack faces with a specified deformation diagram. The physical nature of such connections and the sizes of the tip regions in which the crack faces interact depend on the type of material.

The tip regions are small compared to the remaining part of the plane. Therefore, they can be mentally removed and replaced by cuts whose faces interact with each under a certain law corresponding to the action of the removed material.

Generally, the action of external power and thermal loads on the plate gives rise to normal $[q_y(x)]$ and tangential $[q_{xy}(x)]$ stresses at the connections between the crack faces. Therefore, the crack faces in the tip regions are subjected to the normal and shear stresses $q_y(x)$ and $q_{xy}(x)$, respectively. These stresses are not known beforehand and are to be determined during the solution of the boundary-value problem of fracture mechanics.

The boundary conditions of the problem are written as

$$\sigma_y - i\tau_{xy} = 0 \quad \text{at} \quad y = 0, \qquad \lambda_1 < x < \lambda_2,$$

$$\sigma_y - i\tau_{xy} = q_y - iq_{xy} \quad \text{at} \quad y = 0, \qquad -l \leqslant x \leqslant \lambda_1 \quad \text{and} \quad \lambda_2 \leqslant x \leqslant l,$$

where the stress state is given by

$$\sigma_y = \sigma_{y_1} + \sigma_{y_0}, \qquad \sigma_x = \sigma_{x_1} + \sigma_{x_0}, \qquad \tau_{xy} = \tau_{xy_1} + \tau_{xy_0}.$$

Here σ_{x_0}, σ_{y_0}, τ_{xy_0} is the solution of the thermoelastic problem for the plane without a crack.

Solution of the Boundary-Value Problem. To obtain the stresses σ_{x_0}, σ_{y_0}, and τ_{xy_0}, we solve the thermoelastic problem for the solid plane. We first find the temperature distribution in the plane. For this, the following boundary-value problem of heat-conduction theory is solved:

$$\frac{\partial T}{\partial t} = a\Delta T,$$

$$T = \begin{cases} T_0, & x, y \in S, \\ 0, & x, y \notin S. \end{cases}$$

Here a is the temperature diffusivity of the plane material and Δ is the Laplacian.

For definiteness, we assume that the regions S_1 and S_2 heated from the side of each crack tip by the heat source are rectangles with sides $2x_k$ and $2y_k$ $(k = 1, 2)$ and the center O_k of the rectangle S_k $(k = 1, 2)$ has the coordinates (L_k, b_k) (see Fig. 1).

The temperature distribution has the form

$$T(x, y, t) = T_1(x, y, t) + T_2(x, y, t),$$

$$T_k(x, y, t) = \frac{T_0}{4}\Big[\operatorname{erf}\Big(\frac{x - L_k + x_k}{2\sqrt{at}}\Big) + \operatorname{erf}\Big(\frac{x_k - L_k - x}{2\sqrt{at}}\Big)\Big]\Big[\operatorname{erf}\Big(\frac{y - b_k + y_k}{2\sqrt{at}}\Big) + \operatorname{erf}\Big(\frac{y_k + b_k - y}{2\sqrt{at}}\Big)\Big],$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-u^2)\, du.$$

The perturbed temperature field caused by the presence of the crack is ignored in determining the temperature field to simplify the problem. In particular, if the regions S_k ($k = 1, 2$) are arranged symmetrically about the abscissa, the perturbed temperature field is absent.

The main relations of the formulated problem must be supplemented by an equation that relates the crack opening displacements and the strain at the connections. Without loss of generality, this equation can be written as follows [4, 5]:

$$(v^+ - v^-) - i(u^+ - u^-) = C_y(x, \sigma) q_y(x) - i C_x(x, \sigma) q_{xy}(x). \tag{1}$$

Here the functions $C_y(x, \sigma)$ and $C_x(x, \sigma)$ can be treated as the effective compliances of the connections that depend on their tension and $\sigma = \sqrt{q_y^2 + q_{xy}^2}$ is the stress vector magnitude at the connections. For constant values of C_y and C_x, we have a linear deformation law in (1). Generally, the deformation law is nonlinear and specified.

The stress–strain state for the infinite plane in the plane problem with a cut along the abscissa is described by two analytical functions $\Phi(z)$ and $\Omega(z)$ [6]:

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}],$$

$$\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)},$$

$$2\mu \frac{\partial}{\partial x}(u + iv) = k_0 \Phi(z) - \overline{\Phi(z)} - \overline{z\Phi'(z)} - \overline{\Psi(z)},$$

$$\Omega(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z).$$

Here $k_0 = 3 - 4\nu$ for plane strains, $k_0 = (3 - \nu)/(1 + \nu)$ for plane stresses, and ν is Poisson's constant of the plane material.

To determine the functions $\Phi(z)$ and $\Omega(z)$, we have the linear conjugation problem [6]

$$[\Phi(x) + \Omega(x)]^+ + [\Phi(x) + \Omega(x)]^- = 2p(x); \tag{2}$$

$$[\Phi(x) - \Omega(x)]^+ - [\Phi(x) - \Omega(x)]^- = 0, \tag{3}$$

where $-l \leqslant x \leqslant l$ (x is the affix of the points of the crack contour with the tip zones),

$$p(x) = \begin{cases} 0 & \text{on the free crack faces;} \\ q_y - iq_{xy} & \text{on the faces of the crack-tip zones.} \end{cases}$$

The solution of the boundary-value problem (2), (3) is sought in the form

$$\Phi(z) = \Phi_0(z) + \Phi_1(z), \qquad \Omega(z) = \Omega_0(z) + \Omega_1(z).$$

Here the potentials $\Phi_0(z)$ and $\Omega_0(z)$ describe the thermoelastic state of the solid plane under the action of the heat source.

The complex potentials $\Phi_1(z)$ and $\Omega_1(z)$ are determined from the boundary conditions (2) and (3). To find the functions $\Phi_1(z)$ and $\Omega_1(z)$, we write boundary conditions (2) and (3) in the form

$$[\Phi_1(x) + \Omega_1(x)]^+ + [\Phi_1(x) + \Omega_1(x)]^- = 2p(x) + 2q_0(x); \tag{4}$$

$$[\Phi_1(x) - \Omega_1(x)]^+ - [\Phi_1(x) - \Omega_1(x)]^- = 0, \tag{5}$$

where $2q_0(x) = -[\Phi_0(x) + \Omega_0(x)]^+ - [\Phi_0(x) + \Omega_0(x)]^- = \sigma_{y_0}(x) - i\tau_{xy_0}$.

As $z \to \infty$, we have $\Phi_0(z) \to 0$, $\Phi(z) \to \Phi_1(z) \to \sigma_0/4$, $\Omega_0(z) \to 0$, and $\Omega(z) \to \Omega_1(z) \to 3\sigma_0/4$.

Solving the thermoelastic problem for the solid plane, we obtain

$$2q_0(x, 0) = \sigma_{y_0}(x, 0) - i\tau_{xy_0}(x, 0).$$

Here $\sigma_{y_0} = \sum\limits_{k=1}^{n} \sigma_{y_{0k}}$ and $\tau_{xy_0} = \sum\limits_{k=1}^{n} \tau_{xy_{0k}}$, where $n = 2$ and

$$\sigma_{y_{0k}} = -\frac{\mu(1+\nu)\alpha T_0}{4\sqrt{\pi}}\Big\{4\sqrt{\pi}A(x,y) + \frac{4}{\sqrt{\pi}}\Big[\arctan\Big(\frac{y-b_k+y_k}{x-L_k+x_k}\Big) + \arctan\Big(\frac{y_k+b_k-y}{x_k+L_k-x}\Big)$$

$$+\arctan\Big(\frac{y_k+b_k-y}{x-L_k+x_k}\Big) + \arctan\Big(\frac{y-b_k+y_k}{x_k+L_k-x}\Big)$$

$$-\int_0^t \frac{1}{\tau\sqrt{a\tau}}\Big[(x-L_k+x_k)\exp\Big(-\frac{(x-L_k+x_k)^2}{4a\tau}\Big) + (x_k+L_k-x)\exp\Big(-\frac{(x_k+L_k-x)^2}{4a\tau}\Big)$$

$$\times\Big[\operatorname{erf}\Big(\frac{y-b_k+y_k}{2\sqrt{a\tau}}\Big) + \operatorname{erf}\Big(\frac{y_k+b_k-y}{2\sqrt{a\tau}}\Big)\Big]\,d\tau\Big\},$$

$$\tau_{xy_{0k}} = -\frac{\mu(1+\nu)\alpha T_0}{2\pi}\Big\{\ln\frac{(x-x_k-L_k)^2+(y-b_k+y_k)^2}{(x-x_k-L_k)^2+(y-y_k-b_k)^2} + \ln\frac{(x-L_k+x_k)^2+(y-y_k-b_k)^2}{(x-L_k+x_k)^2+(y-b_k+y_k)^2}$$

$$-\int_0^t \frac{1}{\tau}\Big[\exp\Big(-\frac{(x-L_k+x_k)^2}{4a\tau}\Big) - \exp\Big(-\frac{(x_k+L_k-x)^2}{4a\tau}\Big)\Big]$$

$$\times\Big[\exp\Big(-\frac{(y-b_k+y_k)^2}{4a\tau}\Big) - \exp\Big(-\frac{(y_k+b_k-y)^2}{4a\tau}\Big)\Big]\,d\tau\Big\};$$

$$A(x,y) = \begin{cases} 1, & x,y \in S_k, \\ 0, & x,y \notin S_k; \end{cases}$$

μ is the shear modulus of the plane material, and α is the linear temperature-expansion coefficient.

The general solutions of the boundary-value problems (4) and (5) have the following form [6]:

$$\Phi_1(z) - \Omega_1(z) = -\sigma_0/2,$$

$$\Phi_1(z) + \Omega_1(z) = \frac{1}{\pi i\sqrt{z^2-l^2}}\int_{-l}^{l}\frac{\sqrt{t^2-l^2}\,[p(t)+q_0(t)]}{t-z}\,dt + \frac{2F(z)}{\sqrt{z^2-l^2}}.$$

Here $F(z) = c_0 z + c_1$, and by the function $(z^2-l^2)^{-1/2}$ is meant the branch that at large $|z|$ has the form

$$(z^2-l^2)^{-1/2} = \frac{1}{z} + \frac{l^3}{2z^3} + \dots\ .$$

Finally, for the complex potentials $\Phi_1(z)$ and $\Omega_1(z)$, we have

$$\Phi_1(z) = \frac{1}{2\pi i\sqrt{z^2-l^2}}\int_{-l}^{l}\frac{\sqrt{t^2-l^2}\,[p(t)+q_0(t)]\,dt}{t-z} + \frac{F(z)}{\sqrt{z^2-l^2}} - \frac{\sigma_0}{4}; \tag{6}$$

$$\Omega_1(z) = \frac{1}{2\pi i\sqrt{z^2-l^2}}\int_{-l}^{l}\frac{\sqrt{t^2-l^2}\,[p(t)+q_0(t)]\,dt}{t-z} + \frac{F(z)}{\sqrt{z^2-l^2}} + \frac{\sigma_0}{4}.$$

To determine the coefficient c_0, it is necessary to expand the function (6) in a series in the powers of z in the neighborhood of the point $|z| \to \infty$ and to compare this expansion with the expression

$$\Phi_1(z) = \sigma_0/4 + O(1/z^2).$$

As a result, we obtain $c_0 = \sigma_0/2$. The constant c_1 is determined from the condition of uniqueness of the displacements [6]:

$$\int_{-l}^{l}[\Phi_1^+(x) - \Phi_1^-(x)]\,dx = 0.$$

To finally determine the complex potentials $\Phi_1(z)$ and $\Omega_1(z)$, it is necessary to find the stresses q_y and q_{xy} at the connections. Using the relations $2\mu\,\partial(u+iv)/\partial x = k_0\Phi(z) - \Omega(\bar{z}) - (z-\bar{z})\overline{\Phi'(z)}$ and the boundary values of the functions $\Phi_1(z)$, $\Omega_1(z)$, and $F(z)$, we obtain the following equality on the segment $|x| \leqslant l$:

$$\Phi_1^+(x) - \Phi_1^-(x) = \frac{2\mu}{1+k_0}\Big[\frac{\partial}{\partial x}(u^+ - u^-) + i\frac{\partial}{\partial x}(v^+ - v^-)\Big]. \tag{7}$$

Using the Sokhotsky–Plemelj formulas [7] and formula (6), we obtain

$$\Phi_1^+(x) - \Phi_1^-(x) = -\frac{i}{\pi\sqrt{l^2-x^2}}\Big[\int_{-l}^{l}\frac{\sqrt{l^2-t^2}\,[p(t)+q_0(t)]\,dt}{t-x} + 2(c_0x+c_1)\Big]. \tag{8}$$

Substituting expression (8) into the left side of Eq. (7), using relation (1), and performing some transformations, we obtain the following system of nonlinear integrodifferential equations for the unknown functions $q_y(x)$ and $q_{xy}(x)$:

$$-\frac{1}{\pi\sqrt{l^2-x^2}}\Big[\int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_y(t)\,dt + \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,\sigma_{y_0}(t)\,dt + 2(c_0x+c_1)\Big] = \frac{2\mu}{1+k_0}\frac{\partial}{\partial x}\,(C_y(x,\sigma)q_y(x)); \tag{9}$$

$$-\frac{1}{\pi\sqrt{l^2-x^2}}\Big[\int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_{xy}(t)\,dt + \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,\tau_{xy_0}(t)\,dt\Big] = \frac{2\mu}{1+k_0}\frac{\partial}{\partial x}\,(C_y(x,\sigma)q_{xy}(x)). \tag{10}$$

We recall that

$$I_1 = \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_y(t)\,dt = \int_{-l}^{\lambda_1}\frac{\sqrt{l^2-t^2}}{t-x}\,q_y(t)\,dt + \int_{\lambda_2}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_y(t)\,dt,$$

$$I_2 = \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_{xy}(t)\,dt = \int_{-l}^{\lambda_1}\frac{\sqrt{l^2-t^2}}{t-x}\,q_{xy}(t)\,dt + \int_{\lambda_2}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_{xy}(t)\,dt.$$

Numerical Solution and Analysis. The formulated problem, as might be expected, is split into two independent problems: Eq. (9) for mode I cracks and Eq. 10 for mode II cracks. Each of these equations is a nonlinear integrodifferential equation with a Cauchy kernel and can be solved only numerically. These equation can be solved using a collocation scheme with an approximation of the unknown functions. For the case of a nonlinear deformation law for the connections, it is reasonable to determine the stresses q_y and q_{xy} at the connections using an iterative scheme similar to the elastic solution method [8].

To avoid solving the integrodifferential equations, we write Eqs. (9) and (10) as

$$-\frac{1+k_0}{2\mu}\int_{-l}^{x} Q_1(x)\,dx = C_y(x,\sigma)q_y(x), \qquad -\frac{1+k_0}{2\mu}\int_{-l}^{x} Q_2(x)\,dx = C_x(x,\sigma)q_{xy}(x). \tag{11}$$

Here

$$Q_1(x) = -\frac{1}{\pi\sqrt{l^2-x^2}}\Big[\int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_y(t)\,dt + \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,\sigma_{y_0}(t)\,dt + 2(c_0x+c_1)\Big],$$

$$Q_2(x) = -\frac{1}{\pi\sqrt{l^2-x^2}}\Big[\int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,q_{xy}(t)\,dt + \int_{-l}^{l}\frac{\sqrt{l^2-t^2}}{t-x}\,\tau_{xy_0}(t)\,dt\Big].$$

Le us divide the segment $(-l, l)$ by M nodal points t_m ($m = 1, 2, \ldots, M$) and require that conditions (11) be satisfied at the nodal points. As a result, instead of each of Eqs. (11), we obtain algebraic systems of M_1 equations for the approximate values of $q_y(t_m)$ and $q_{xy}(t_m)$ ($m = 1, 2, \ldots, M_1$), respectively:

$$
\begin{gathered}
CQ_1(t_1) = C_y(t_1)q_y(t_1),\\
C(Q_1(t_1) + Q_1(t_2)) = C_y(t_2)q_y(t_2),\\
\dots\dots\\
C\sum_{m=1}^{M_1} Q_1(t_m) = C_y(t_{M_1})q_y(t_{M_1});
\end{gathered}
\tag{12}
$$

$$
\begin{gathered}
CQ_2(t_1) = C_x(t_1)q_{xy}(t_1),\\
C(Q_2(t_1) + Q_2(t_2)) = C_x(t_2)q_{xy}(t_2),\\
\dots\dots\\
C\sum_{m=1}^{M_1} Q_2(t_m) = C_x(t_{M_1})q_{xy}(t_{M_1}).
\end{gathered}
\tag{13}
$$

Here $C = -\dfrac{1+k_0}{2\mu}\,\dfrac{\pi l}{M}$ and M_1 is the number of nodes belonging to the tip zones of the crack.

In obtaining the algebraic systems, all intervals of integration were reduced to one interval $[-1, 1]$, and the integrals were then replaced by finite sums using quadrature formulas of the type of the Gauss distribution.

In the particular case of linearly elastic connections, systems (12) and (13) are linear and were solved numerically using the Gauss method with a choice of the basic element. After the solution of the algebraic systems (12) and (13), the stress intensity factors were calculated.

According to the superposition principle, in the case of connections (adhesive forces) in the crack-tip zone, the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ are conveniently written as

$$
K_{\rm I} - iK_{\rm II} = (K_{\rm I}^{\rm load} + K_{\rm I}^{\rm c}) - i(K_{\rm II}^{\rm load} + K_{\rm II}^{\rm c}),
\tag{14}
$$

where $K_{\rm I}^{\rm load}$ and $K_{\rm II}^{\rm load}$ are the stress intensity factors due to the power and thermal loads and $K_{\rm I}^{\rm c}$ and $K_{\rm II}^{\rm c}$ are the stress intensity factors due to the stresses arising in the crack-tip zone.

Using well-known formulas [9], for the left tip of the crack we obtain

$$
\begin{gathered}
K_{\rm I}^{\rm load} = \sigma_0\sqrt{\pi l} + \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} \sigma_{y_0}(x)\sqrt{\frac{l-x}{x+l}}\,dx, \qquad K_{\rm I}^{\rm c} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} q_y(x)\sqrt{\frac{l-x}{x+l}}\,dx,\\
K_{\rm II}^{\rm load} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} \tau_{xy_0}(x)\sqrt{\frac{l-x}{x+l}}\,dx, \qquad K_{\rm II}^{\rm c} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} q_{xy}(x)\sqrt{\frac{l-x}{x+l}}\,dx.
\end{gathered}
\tag{15}
$$

Similarly, for the right tip of the crack we have

$$
\begin{gathered}
K_{\rm I}^{\rm load} = \sigma_0\sqrt{\pi l} + \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} \sigma_{y_0}(x)\sqrt{\frac{x+l}{l-x}}\,dx, \qquad K_{\rm I}^{\rm c} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} q_y(x)\sqrt{\frac{x+l}{l-x}}\,dx,\\
K_{\rm II}^{\rm load} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} \tau_{xy_0}(x)\sqrt{\frac{x+l}{l-x}}\,dx, \qquad K_{\rm II}^{\rm c} = \frac{1}{\sqrt{\pi l}}\int_{-l}^{l} q_{xy}(x)\sqrt{\frac{x+l}{l-x}}\,dx.
\end{gathered}
\tag{16}
$$

Le us consider the energy characteristics for the crack with connections between the faces. Irrespective of the form of the deformation law for the connections, the rate of strain energy release is defined by the relation [2, 9]

$$
G_{\rm rel} = (1-\nu)K_{\rm con}^2/(2\mu),
\tag{17}
$$

where $K_{\rm con} = \sqrt{K_{\rm I}^2 + K_{\rm II}^2}$ is the modulus of the stress intensity factors in the presence of connections in the tip zone of the crack.

The rate of consumption of the strain energy by the connections in the tip zone of the crack is given by

$$G_n = \frac{1}{b}\frac{\partial U_n}{\partial l} \qquad \Big(U_n = b\int\limits_{\lambda_k}^{l} f(u)\,dx, \qquad f(u) = \int\limits_0^{v(x)} q_y(v)\,dv + \int\limits_0^{u(x)} q_{xy}(u)\,du\Big). \tag{18}$$

Here b is the thickness of the plane, U_n is the work of deformation of the connections, and $f(u)$ is the strain energy density at the connections in the tip zone of the crack.

Using (18) and taking into account that

$$v^+(l) - v^-(l) = 0, \qquad v^+(l) - v^-(l) = 0,$$

we obtain [5]

$$G_n = b\int\limits_{l-\lambda_k}^{l} \frac{\partial}{\partial l}(v^+ - v^-)q_y(x)\,dx + b\int\limits_{l-\lambda_k}^{l} \frac{\partial}{\partial l}(u^+ - u^-)q_{xy}(x)\,dx. \tag{19}$$

As is known, for the limiting equilibrium state, the following condition is satisfied:

$$G_{\rm rel} = G_{\rm lim}. \tag{20}$$

Condition (20) is a necessary but insufficient condition for the limiting equilibrium state of the crack with the tip zone. Therefore, to determine the limiting-equilibrium state of the crack tip and the tip zone, it is necessary to introduce an additional critical condition. As such an additional condition, we use the critical crack opening. We assume that rupture of the connections at the edge of the tip zone ($x_0 = \lambda_k$) occurs if the following condition is satisfied:

$$V(x_0) = \sqrt{v^2(x_0) + u^2(x_0)} = \delta_k. \tag{21}$$

Here δ_k is the limiting stretching (length) of the connections, $v = v^+ - v^-$, and $u = u^+ - u^-$. Simultaneous solution of Eqs. (20) and (21) (for specified crack length and characteristics of the connections) yields the critical external load and the size of the tip zone $d_k = l - |\lambda_k|$ for the limiting equilibrium state of the crack tip and the edge of the tip zone. The rate of consumption of the strain energy $G_k(d_k, l)$ obtained from this solution is an energy characteristic of the crack strength, i.e., $G_k = G_n(d_k, l)$.

From the aforesaid, using the limiting values of δ_k and G_k for the specified sizes of the crack and the tip zone, it is possible to distinguish the regimes of crack equilibrium and growth under monotonic loading.

If the conditions $G_{\rm rel} \geqslant G_k$ and $V(x_0) < \delta_k$ are satisfied for the specified size of the tip zone, advance of the crack tip occurs with a simultaneous increase in the length of the tip zone without rupture of the connections.

This stage of crack propagation can be regarded as a process of adaptation to the specified level of external loads. Growth of the crack tip with simultaneous rupture of the connections at the edge of the tip zone occurs if the conditions $G_{\rm rel} \geqslant G_k$ and $V(x_0) \geqslant \delta_k$ are satisfied. Thus, for example, if the inequalities $G_{\rm rel} < G_k$ and $V(x_0) \geqslant \delta_k$ are satisfied, the connections are ruptured without advance of the crack tip and the size of the tip zone reduces, tending to the critical value for the specified load level.

Finally, if the conditions $G_{\rm rel} < G_k$ and $V(x_0) < \delta_k$ are satisfied, the position of the crack tip and the tip zone do not change.

Thus, the analysis shows that values of the external load and the critical parameters δ_k and G_k determine the nature of the fracture, namely:

— growth of the crack tip with advance of the tip zone;

— reduction in the size of the tip zone without growth of the crack tip;

— growth of the crack tip with simultaneous rupture of the connections at the edge of the tip zone.

In the case of a nonlinear deformation law for the connections, the stresses in the tip zones are determined using an iterative algorithm similar to the elastic solution method [8].

It is assumed that the law of deformation of the interparticle connections (adhesive forces) is linear for $V \leqslant V_*$.

The first step of the iterative process consists of solving system (12), (13) for linearly elastic interparticle connections. Subsequent iterations are performed only if the relation $V(x) > V_*$ holds on part of the tip zone.

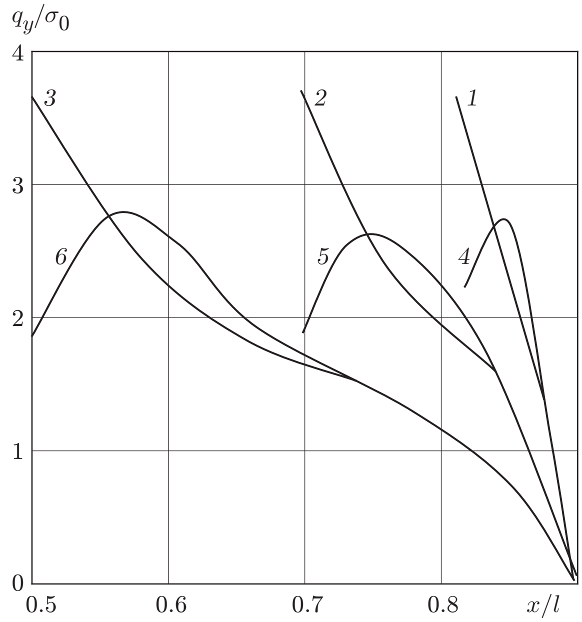

Fig. 2

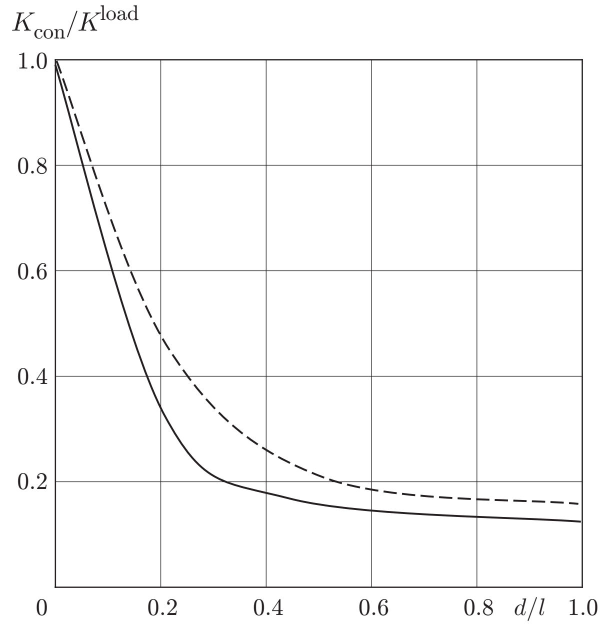

Fig. 3

Fig. 2. Distribution of normal stresses at the connections of the crack-tip zones: curves 1–3 refer to a linear connection and curves 4–6 refer to a bilinear connection; $d/l = 0.15$ (4), 0.3 (2 and 5), 0.5 (3 and 6).

Fig. 3. Relative modulus of the stress intensity factors versus the size of the crack-tip zone.

For such iterations, we solve the system of equations in each approximation for quasielastic connections with an effective compliance which varies along the tip zone of the crack and depends on the stress vector magnitude at the connections obtained in the previous step of the calculation. The effective compliance is calculated in a similar manner as the secant modulus in the method of variable elasticity parameters [10]. The process of successive approximations is terminated when the stresses along the tip zone obtained in two series iterations differ from each other only slightly.

The nonlinear part of the strain curve for the connections is represented as a bilinear dependence [5] whose ascending segment corresponds to the elastic deformation of the connections ($0 < V(x) \leqslant V_*$) with their maximum tension. For $V(x) > V_*$, the deformation law is described by the nonlinear dependence specified by the points (V_*, σ_*) and (δ_k, σ_k). For $\sigma_k \geqslant \sigma_*$, we have an increasing linear dependence (linear strengthening corresponding to the elastoplastic deformation of the connections).

Thus, the bilinear dependence between the tension of the connection $\sigma(x)$ and its stretching $V(x)$ is represented [5] in the form

$$\sigma(V) = \begin{cases} V(x)/C(x), & 0 \leqslant V(x) \leqslant V_*, \\ \sigma_k + (\sigma_* - \sigma_k)(\delta_k - V(x))/(\delta_k - V_*), & V_* < V(x) \leqslant \delta_k, \end{cases}$$

where $C(x) = C_y(x) = C_x(x)$ is the effective compliance of the connections at the point with coordinate x in the tip region. Obviously, if the compliances of the elastic connections vary along the tip zone of the crack, the effective compliance $C(x)$ is also a variable, which corresponds to variation of the law of deformation of the connections along the tip zone of the crack.

Figure 2 shows the distribution of normal stresses at the connections in the crack-tip zones for the following values of the free parameters:

$$t_* = 4at/L_1^2 = 10, \qquad x_1/L_1 = 0.75, \qquad y_1/L_1 = 0.5, \qquad b_1/L_1 = 0.2;$$

$$\nu = 0.3, \qquad x_2/L_2 = 0.7, \qquad y_2/L_2 = 0.6, \qquad b_2/L_2 = 0.3, \qquad L_1 = L_2.$$

The calculations show that the presence of the temperature stresses induced by the heat source reduces the stress intensity factors, the stresses at the connections between the faces, and the crack opening. For a linear deformation law for the connections, the stresses in them always have maximum values at the edge of the tip zone. A similar picture is observed for the crack opening value; i.e., at the edge of the tip zone, it is maximum for linear and nonlinear deformation laws, and with increase in the relative compliance of the connections, the crack opening increases.

Figure 3 gives curves of the relative modulus of the stress intensity factors $K_0 = K_{\rm con}/K^{\rm load}$ (which can be treated as a strengthening factor; $K^{\rm load} = \sqrt{(K_{\rm I}^{\rm load})^2 + (K_{\rm II}^{\rm load})^2}$) versus the size of the crack-tip zone in the plane. Here the solid curve corresponds to the right tip of the crack, and the dashed curve to the left tip of the crack. The calculations show that as the relative compliance decreases, the strengthening factor also decreases.

For the case of a crack with connections in the tip zones and temperature stresses induced by heat sources, an analysis of the limiting equilibrium state of the plane reduces to a parametric study of the solution of the algebraic systems (12) and (13) for various laws of deformation of the connections, various sizes of the crack-tip regions, and various thermal and elastic constants of the plane material. The normal and tangential stresses at the connections and the crack opening are determined directly by solving the resulting algebraic systems in each approximation. The crack opening in the tip zones can also be determined from relation (1). The stress intensity factors and the rates of energy release and absorption are calculated from formulas (14)–(16) and (17) and (18), respectively.

Following [11] and other studies, it can be assumed that the forces of interaction of the crack faces (adhesive forces) are distributed so that the total stress intensity factor, determined as the difference between the stress intensity factors due to the external and thermal loads and the stress intensity factor due to the adhesive forces applied to the tip zones, is equal to zero. For this model, the problem is solved using the above computational scheme but the sizes of the tip zones with connections are not known beforehand and are to be determined. This is done using the postulate that the singularities in the stress distribution are eliminated, i.e., that the total stress intensity factor is equal to zero. Thus, the main constitutive equations are supplemented by the condition that the stresses in the neighborhood of each crack tip are finite.

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